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Actions of Hopf-Ore Extensions on Path Algebras

QuaSy-Con II

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Acknowledgements

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Outline

Motivating Example

The classical notion of symmetry can be encoded using the language of group actions.

The structure of a Hopf algebra encodes a generalized notion of symmetry, including quantum symmetries.

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The structure of a Hopf algebra encodes a generalized notion of symmetry, including quantum symmetries.

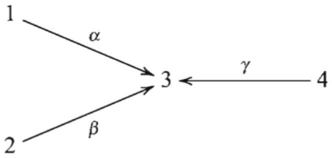
Goal: Study how certain Hopf-Ore extensions act on path algebras of quivers.

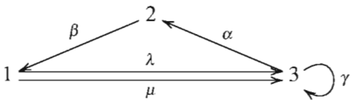
We first recall some background material about quivers.

Definition (Quiver)

A **quiver**, $Q = (Q_0, Q_1, s, t)$ consists of a set of vertices, Q_0, a set of arrows, Q_1, a map mapping an arrow to its starting point $s : Q_1 \longrightarrow Q_0$, and a map mapping an arrow to its terminal point $t : Q_1 \longrightarrow Q_0$.

Examples:





Definition (Path Algebra, [Sch14])

Let Q be a quiver. The **path algebra** $\Bbbk Q$ of Q is the algebra with basis the set of all paths in the quiver Q and with multiplication defined on two basis elements p, p' by

$$pp' = \begin{cases} p \cdot p' & \text{if } s(p') = t(p) \\ 0 & \text{otherwise.} \end{cases}$$

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- For all $i \in Q_0$, e_i denotes the path of length 0 at i in $\Bbbk Q$.
- Moreover, we have the relations $e_i e_j = \delta_{i,j} e_i$ for all $i \in Q_0$ and $e_{sa} a = a = a e_{ta}$ for all $a \in Q_1$.

To classify actions of Hopf algebras on path algebras, we use properties of $\Bbbk Q$.

Nice Properties of $\Bbbk Q$:

- $\Bbbk Q$ is graded, $\Bbbk Q = \oplus_{l=0}^{\infty} \Bbbk Q_l$.
- $\Bbbk Q \cong T_{\Bbbk Q_0}(\Bbbk Q_1)$
  $= \Bbbk Q_0 \oplus \Bbbk Q_1 \oplus (\Bbbk Q_1 \otimes \Bbbk Q_1) \oplus \ldots$.

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Hence we only need consider how these Hopf algebras act on Q_0 and Q_1.

Motivation for Hopf-Ore Extensions

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We first considered Noetherian Hopf algebras of GK dimension 1 and 2.

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$U_q(\mathfrak{b})$	$x, g^{\pm 1}$	$gx = qxg$	$\Delta(x) = x \otimes g + 1 \otimes x$
$H(n, t, q)$	$x, g^{\pm 1}$	$gx = q^m xg$	$\Delta(x) = x \otimes g^t + 1 \otimes x$
$B(n, w, q)$	$x, g^{\pm 1}, h^{\pm 1}$	$gx = xg,\ xh = qhx,$ $x^n = 1 - g^w = 1 - h^w$	$\Delta(x) = x \otimes h + 1 \otimes x$
$C(n, q)$	$x, g^{\pm 1}$	$xg = qgx + g^n - g$	$\Delta(x) = x \otimes g^{n-1} + 1 \otimes x$

Table: Various such Hopf Algebras [Goo12]

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Question: Is there a type of Hopf algebra that encompasses these examples?

Ore Extensions

Let A be a $\Bbbk$-algebra.

Definition

Let τ be an endomorphism of A. A τ-**derivation** of A is any additive map $\delta : A \longrightarrow A$ such that $\delta(ab) = \tau(a)\delta(b) + \delta(a)b$ for all $a, b \in A$ and $\delta(\Bbbk) = 0$.

The **Ore extension**, $R = A[x; \tau, \delta]$, of the $\Bbbk$-algebra A is the $\Bbbk$-algebra R generated by the variable x and the algebra A with the relation that for all $a \in A$

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Question: Assuming A is a Hopf algebra, when is R a Hopf algebra?

Hopf-Ore Extensions

Definition

Let A and $R = A[x; \tau, \delta]$ be Hopf algebras over $\Bbbk$. The Hopf algebra $R = A[x; \tau, \delta]$ is a **Hopf-Ore extension** if $\Delta(x) = x \otimes h + 1 \otimes x$ for some $h \in A$ and A is a Hopf subalgebra of R.

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- If A is cocommutative, for $\alpha : A \to \Bbbk$ with $\alpha(uv) = \alpha(u)\varepsilon(v) + \chi(u)\alpha(v)$,

$$\delta(a) := \alpha(a_1)(1-h)a_2.$$

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Hopf-Ore Extensions of Group Algebras

Let $A = \Bbbk G$ for some group G and consider the Hopf-Ore Extension, $R = A[x; \tau, \delta]$. For some $h \in Z(\Bbbk G)$ and for all $g \in G$,

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Proposition ([Pan03])

Every Hopf-Ore extension of $\Bbbk G$ is of the form above.

Hopf Action

Goal: Study how Hopf-Ore extensions act on path algebras of quivers.

Definition

Let H be a Hopf algebra and an algebra A.
A **(left) Hopf action** of H on A consists of a left H-module structure on A satisfying:

1. $h \cdot (pq) = \sum_i (h_{i,1} \cdot p)(h_{i,2} \cdot q)$ for all $h \in H$ and $p, q \in A$ where $\Delta(h) = \sum_i h_{i,1} \otimes h_{i,2}$ and
2. $h \cdot 1_A = \varepsilon(h) 1_A$ for all $h \in H$.

Example: Let $e_i, e_j \in Q_0$ and $x \in H$ such that $\Delta(x) = x \otimes h + 1 \otimes x$ for $h \in G(H)$. Then

$$x \cdot (e_i e_j) = (x \cdot e_i)(h \cdot e_j) + e_i(x \cdot e_j).$$

Let $R = \Bbbk G(\chi, h, \delta)$ be a Hopf-Ore extension of the group algebra, $\Bbbk G$, as described. Let Q be a quiver with path algebra $\Bbbk Q$ and vertex set Q_0.

Theorem (joint work with Kinser)

1. *The following data determines a Hopf action of R on $\Bbbk Q_0$.*
   - **1.1** *A permutation action of G on the set Q_0;*
   - **1.2** *A collection of scalars $(\gamma_i \in \Bbbk)_{i \in Q_0}$ such that*

$$\gamma_{g \cdot i} = \chi(g)\gamma_i + \alpha(g) \text{ for all } i \in Q_0 \text{ and for all } g \in G.$$

   The x-action is given by

$$x \cdot e_i = \gamma_i e_i - (\chi(h)\gamma_i + \alpha(h))e_{h \cdot i} \qquad \text{for all } i \in Q_0.$$

2. *Every action of R on $\Bbbk Q_0$ is of the form above.*

Theorem (joint work with Kinser)

1. *The following data determines a Hopf action of* $R = \Bbbk G(\chi, h, \delta)$ *on* $\Bbbk Q$.
   1.1 *A Hopf action of* R *on* $\Bbbk Q_0$;
   1.2 *A representation of* G *on* $\Bbbk Q_1$ *satisfying* $s(g \cdot a) = g \cdot sa$ *and* $t(g \cdot a) = g \cdot ta$ *for all* $a \in Q_1$ *and all* $g \in G$;
   1.3 *A* $\Bbbk$*-linear endomorphism* $\sigma : \Bbbk Q_0 \oplus \Bbbk Q_1 \longrightarrow \Bbbk Q_0 \oplus \Bbbk Q_1$ *satisfying some technical conditions. With this data, the* x*-action on* $a \in Q_1$ *is given by*

$$x \cdot a = \gamma_{ta} a - (\chi(h)\gamma_{sa} + \alpha(h))(h \cdot a) + \sigma(a).$$

2. *Every (filtered) action of* R *on* $\Bbbk Q$ *is of the form above.*

Applications to Specific Hopf Algebras

This work allows us to recover the action of $U_q(\mathfrak{b})$ on $\Bbbk Q$ as described in Proposition 3.1 and Theorem 3.13 of [KO21].

Next, we consider a specific Noetherian Hopf algebra of GK dimension 2;

Consider the Hopf algebra $C(n, q)$.

Definition ($C(n, q)$, [Goo89])

Given $n \in \mathbb{Z}$ and $q \in \Bbbk^{\times}$, let $C(n, q)$ be the $\Bbbk$-algebra given by generators $g^{\pm 1}$ and x subject to the relation

$$xg = q^r gx + g^n - g.$$

The unique Hopf algebra structure on $C(n, q)$ is given by

$$\Delta(g) = g \otimes g \text{ and } \Delta(x) = x \otimes g^{n-1} + 1 \otimes x.$$

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$\chi(g) = q^r, h = g^{n-1}$, and $\alpha(g)(1 - g^{n-1})g = g^n - g$ so $\alpha(g) = -1$

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$$\gamma_{g \cdot i} = q^r \gamma_i - 1 \text{ for all } i \in Q_0.$$

   The x*-action is given by*

$$x \cdot e_i = \gamma_i e_i + (1 + q^r + q^{2r} + \cdots + q^{r(n-2)})\gamma_i e_{g^{n-1} \cdot i} \qquad \text{for all } i \in Q_0.$$

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$$x \cdot a = \gamma_{ta} a + (1 + q^r + q^{2r} + \cdots + q^{r(n-2)})\gamma_{sa}(g^{n-1} \cdot a) + \sigma(a).$$

2. *Every (filtered) action of* $C(n,q)$ *on* $\Bbbk Q$ *is of the form above.*

Future Directions

- Classify the actions of more general Hopf-Ore extensions, $R = A[x; \tau, \delta]$ where the comultiplication is of the form:

$$\Delta(x) = x \otimes a + b \otimes x + v(x \otimes x) + w$$

for $a, b \in A$ and $v, w \in A \otimes A$.

- Classify the indecomposable $\Bbbk Q_0$-bimodules in $\mathrm{rep}(\Bbbk G(\chi, h, \delta))$.

Thank you!

Are there any questions?

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